**Scientific Report on the implementation of the project**
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Time optimal controllability for infinite dimensional systems

In the frame of the present project the following activities took place: documentation, updating the bibliography, conferences, scientific contacts, analysis, research and publication of the results. We mention that 4 articles in ISI journals were foreseen to be publised and **we have published 5 articles in ISI journals**, we have submitted for publication another one in an ISI journal and another paper is in preparation.

The research objectives for the mentioned period of time are:

**Stage 1**. To establish new regularity results for the cost functions associated to a linear system.

**Stage 2**. To prove the equivalence between the minimum time problem and the minimum norm control associated to a linear system.

**Stage 3.** To study variational problems for optimal control problems.

All these research objectives were realized. In the following we give a presentation of the main results obtained.

**Stage 1**. To establish new regularity results for the cost functions associated to a linear system.

Let $X$ and $U$ be two Banach spaces and consider the control system represented by

$$y\,(t, x, u) = S\,(t)\,x + V\,(t)\,u,\ \ t > 0 \tag{1}$$

$$y\,(0, x, u) = x,$$

where $y$ is the state, $t$ the time and $u$ the control. Here, $\{S(t);\ t \geq 0\}$ is a $C_0$-semigroup on $X$ and $\{V\,(t)\,;\ t > 0\}$ is a family of bounded linear operators, $V\,(t) : L^\infty\,(0, t; U) \to X$, such that the following condition is satisfied

$$V\,(t_1 + t_2)\,u = S\,(t_2)\,V\,(t_1)\,u + V\,(t_2)\,J_{t_1}u, \tag{2}$$

for all $t_1, t_2 > 0$ and $u \in L^\infty(0, t_1 + t_2; U)$, where $J_{t_1}$ is a translation operator given by

$$(J_{t_1}u)\,(s) = u\,(s + t_1) \tag{3}$$

for $s \geq 0$. Clearly, in $V\,(t_1)\,u$ we have considered the restriction of $u$ to $[0, t_1]$. Further, assume that for each $u \in L^\infty(0, +\infty; U)$ we have that $t \mapsto V(t)u$ is continuous and $\lim_{t \to 0} V(t)u = 0$.

The typical example is the distributed control system

$$y'(t) = Ay(t) + Bu(t), \tag{4}$$

where $A$ is the generator of $\{S(t);\ t \geq 0\}$ and $B$ is linear and bounded from $U$ to $X$. In this case, $V(t)u = \int_0^t S(t-s)Bu(s)ds$. The operator $B$ could be also unbounded to cover the boundary control systems.

For $r > 0$ and $t > 0$ define

$$\mathcal{U}_r\,(t) = \{u \in L^\infty(0, t; U);\ \ \|u\|_\infty \leq r\}.$$

Denote by $\mathcal{C}_r\,(t)$ the null controllable set at time $t > 0$, i.e., the set of all points $x \in X$ for which there exists $u \in \mathcal{U}_r\,(t)$ with $y(t, x, u) = 0$. Consider $\mathcal{C}_r\,(0) = \{0\}$ and set $\mathcal{C}_r = \bigcup_{t \geq 0} \mathcal{C}_r(t)$, called the null controllable set, and define the minimum time function $\mathcal{T} : (0, +\infty) \times X \to [0, +\infty]$ by

$$\mathcal{T}(r, x) = \begin{cases} \inf\{t \geq 0;\ x \in \mathcal{C}_r(t)\},\ \text{if } x \in \mathcal{C}_r \\ +\infty,\ \text{elsewhere.} \end{cases}$$

For $t>0$ and $x \in X$, denote by $\mathcal{M}(t,x)$ the (possibly empty) set of controls $u \in L^{\infty}(0,t;U)$ such that $y(t,x,u)=0$ and define the control cost to bring $x$ to 0 as the function $\mathcal{E}:(0,+\infty)\times X \to [0,+\infty]$ given by

$$\mathcal{E}(t,x)=\inf_{u\in\mathcal{M}(t,x)} \|u\|_{\infty}.$$

The basic hypothesis we shall refer to in the sequel is the following.

**(H)** There exists $\gamma:(0,+\infty)\to(0,+\infty)$ such that

$$S(t)B(0,\gamma(t)) \subseteq V(t)B(0,1), \tag{5}$$

for any $t>0$. Here, $B(0,\gamma(t))$ is the closed ball of radius $\gamma(t)$ from $X$, while $B(0,1)$ is the closed unit ball from $L^{\infty}(0,t;U)$, i.e., $\mathcal{U}_1(t)$.

By the open mapping theorem, (5) is equivalent to

$$Range\,(S\,(t)) \subseteq Range\,(V\,(t))\,,$$

which means that all points of $X$ can be transferred to zero in time $t$ by $L^{\infty}\,(0,t;U)$-controls.

We further state some additional hypotheses that we shall frequently use in the sequel.

**(H1)** $X$ and $U$ are reflexive Banach spaces.

**(H2)** For every $t>0$, $V(t)=H(t)^{*}$ for some $H(t):X^{*}\to L^{1}(0,t;U^{*})$.

It is well known that for every $C_0$-semigroup $\{S\,(t)\,;\ t\geq 0\}$ there exist $M\geq 1$ and $\omega\in\mathbb{R}$ such that $\|S(t)\|\leq Me^{\omega t}$, for any $t\geq 0$.

**Proposition 0.0.1** *Assume **(H)**.*
*(i) For any $r>0$, $x\in X$ and $t>0$, there exists $u^{*}\in\mathcal{U}_r\,(t)$ such that*

$$\|y(t,x,u^{*})\| \leq Me^{\omega t}\,|\|x\|-r\gamma(t)|\,. \tag{6}$$

*(ii) If $\|x\|\leq r\gamma(t)$, then $x\in\mathcal{C}_r(t)$.*
*(iii) If $x_0\in\mathcal{C}_r\,(t)$ and $\|x-x_0\|\leq\rho\gamma\,(t')$ for some $t'\in(0,t]$ and $\rho>0$, then $x\in\mathcal{C}_{r+\rho}\,(t)$.*
*(iv) Let $0<r_1<r_2$. If $x\in\mathcal{C}_{r_1}\,(t)$, then there exists $s\in(0,t)$ such that $x\in\mathcal{C}_{r_2}\,(s)$.*
*(v) Let $0<t_1<t_2$. If $x\in\mathcal{C}_r\,(t_1)$, then there exists $\overline{r}\in(0,r)$ such that $x\in\mathcal{C}_{\overline{r}}\,(t_2)$.*

Further, we give estimates of the minimum time function around points in the null controllable set and around points in the boundary of the null controllable set. Moreover, we get local uniform continuity of the minimum time function on the null controllable set.

In what follows we denote

$$M_{\gamma}:=\sup_{s\in\mathbb{R}_+}\gamma(s)\in(0,+\infty].$$

**Theorem 1** *Assume **(H)**. Let $r>0$. Then, for any $x\in X$ with $\|x\|<rM_{\gamma}$ we have $x\in\mathcal{C}_r$. Assume further that the function $\gamma$ in (5) is continuous, strictly increasing and $\gamma\,(0)=0$.*
*(i) For any for any $x\in X$ with $\|x\|<rM_{\gamma}$, $\mathcal{T}(r,x)\leq\gamma^{-1}\,(\|x\|\,/r)$.*
*(ii) Let $x\in\mathcal{C}_r$ and $z\in X$ such that $\|x-z\|<(r/M)\,e^{-\omega\mathcal{T}(r,x)}M_{\gamma}$. Then $z\in\mathcal{C}_r$ and*

$$\mathcal{T}(r,z)\leq\mathcal{T}(r,x)+\gamma^{-1}\left(\frac{\|x-z\|}{r}Me^{\omega\mathcal{T}(r,x)}\right). \tag{7}$$

*(iii) In the case $\omega>0$, if $x\in\mathcal{C}_r$ and $z\notin\mathcal{C}_r$, then*

$$\mathcal{T}\,(r,x)\geq-\frac{1}{\omega}\log\left(\frac{\|x-z\|\,M}{rM_{\gamma}}\right).$$

*Consequently,* $\lim_{x\to z}\mathcal{T}(r,x)=+\infty$, *for any* $z\in\partial\mathcal{C}_r$.
*In the case* $\omega\le 0$ *we have that* $X=\mathcal{C}_r$.
*(iv) If* $M_\gamma<+\infty$, *then* $\mathcal{C}_r$ *is open and for any* $x_0\in\mathcal{C}_r$ *we have*

$$|\mathcal{T}(r,z_1)-\mathcal{T}(r,z_2)|\le\gamma^{-1}\left(c_r\|z_1-z_2\|\right)\tag{8}$$

*for any* $z_1,z_2\in B(x_0,\delta_r)$, *where, in the case* $\omega>0$,

$$c_r=\frac{M}{r}e^{\omega\left(\mathcal{T}(r,x_0)+\gamma^{-1}(M_\gamma/k)\right)}\text{ and }\delta_r=\frac{rM_\gamma}{Mk}e^{-\omega(\mathcal{T}(r,x_0)+M_\gamma)}\tag{9}$$

*for some* $k>\max\{M_\gamma/\gamma(M_\gamma),2\}$ *and, in the case* $\omega\le 0$,

$$c_r=\frac{M}{r}\text{ and }\delta_r=\frac{rM_\gamma}{2M}.\tag{10}$$

*(v) If* $M_\gamma=+\infty$, *then* $\mathcal{C}_r=X$ *and, in the case* $\omega>0$, *for any* $x_0\in X$ *and any* $\delta>0$ *there exists*

$$c_r=\frac{M}{r}e^{\omega\left(\mathcal{T}(r,x_0)+\gamma^{-1}(M\delta/re^{\omega\mathcal{T}(r,x_0)})\right)}\tag{11}$$

*such that* (8) *holds for any* $z_1,z_2\in B(x_0,\delta)$. *In the case* $\omega\le 0$, (8) *holds for any* $z_1,z_2\in X$ *where* $c_r=M/r$.

One of the main results obtained is the continuity of the minimum time function $\mathcal{T}$, as a function of both variables.

**Theorem 2** *Assume the hypotheses of Theorem* 1 *and* ***(H1)***. *Let* $r_0>0$ *and* $x_0\in X$ *be such that* $x_0\in\mathcal{C}_{r_0}$. *Then the minimum time function* $\mathcal{T}$ *is continuous in* $(r_0,x_0)$.

We prove the Lipschitz continuity of the minimum energy function in the state variable.

**Proposition 0.0.2** *Assume* ***(H)***. *Then, for every* $x,z\in X$ *and every* $t>0$, *we have*

$$|\mathcal{E}(t,x)-\mathcal{E}(t,z)|\le\frac{1}{\gamma(t)}\|x-z\|.$$

**Stage 2**. To prove the equivalence between the minimum time problem and the minimum norm control associated to a linear system.

In the following we give a result on the equivalence between the minimum time function and the minimum energy and on the monotonicity of these two functions. The proof follows using the estimates obtained in Proposition 0.0.1.

**Theorem 3** *Suppose the existence of optimal controls for the minimum time and minimum energy problems. Assume* ***(H)***. *Then,*
*(i) if* $0<t_1<t_2$ *and* $\mathcal{E}(t_1,x)>0$, *then*

$$\mathcal{E}(t_2,x)<\mathcal{E}(t_1,x);$$

*(ii) if* $0<r_1<r_2$ *and* $x\in\mathcal{C}_{r_1}$, *then*

$$\mathcal{T}(r_2,x)<\mathcal{T}(r_1,x);$$

*(iii) for any* $r>0$ *and* $x\in\mathcal{C}_r$ *we have*

$$\mathcal{E}(\mathcal{T}(r,x),x)=r;\tag{12}$$

*(iv) for any* $t>0$ *and* $x\in X$ *we have*

$$\mathcal{T}(\mathcal{E}(t,x),x)=t.$$

**Remark 0.0.1** *Theorem 3 says that, for $r>0$ and $x\in\mathcal{C}_r$, $\mathcal{T}(r,x)$ is the unique solution of the equation $\mathcal{E}(t,x)=r$. Also, given $t>0$ and $x\in X$, $\mathcal{E}(t,x)$ is the unique solution of $\mathcal{T}(r,x)=t$.*

From Theorem 3 we easily get the following result.

**Corollary 0.0.1** *Suppose the hypotheses of Theorem 3. Let $t>0$ and $u\in L^{\infty}(0,t;U)$ a minimum norm control such that $y(t,x,u)=0$. Then $u$ is a time optimal control for $x$, under the norm constraint $r=\mathcal{E}(t,x)$. Conversely, let $r>0$ and $v\in\mathcal{U}_r(\mathcal{T}(r,x))$ a time optimal control for $x$. Then $v$ is a minimum norm control on $[0,\mathcal{T}(r,x)]$.*

**Remark 0.0.2** *Since $\mathcal{E}(\cdot,x)$ and $\mathcal{T}(\cdot,x)$ are inverse one to another, under the hypotheses of Proposition 2 , we get that $\mathcal{E}(\cdot,x)$ is continuous on $(0,+\infty)$. Let us consider the function*

$$\mathcal{E}(t)=\sup_{\|x\|\leq 1}\mathcal{E}(t,x),\ t>0,$$

*which is lower semicontinuous. Then, we can get a function $\gamma^*$ satisfying (5) which is upper semicontinuous. Indeed, define*

$$\gamma^*(t)=\sup\{\gamma(t);\ \gamma \text{ satisfies } (5)\}.$$

*It is easy to prove that*

$$\mathcal{E}(t)=\frac{1}{\gamma^*(t)},$$

*hence $\gamma^*$ is upper semicontinuous.*

Moreover, we studied two main situations when the limit of Pareto minima of a sequence of perturbations of a set-valued map $F$ is a critical point of $F$. The concept of criticality is understood in the Fermat generalized sense by means of limiting (Mordukhovich) coderivative. Firstly, we consider perturbations of enlargement type which, in particular, cover the case of perturbation with dilating cones. Secondly, we study the case of Aubin type perturbations, and for this we introduce and study a new concept of openness with respect to a cone.

We briefly present next the main results.

Let $K\subset Y$ be a convex closed cone and, additionally, we suppose it is as well pointed (that is, $K\cap -K=\{0\}$) and proper (that is, $K\neq\{0\}$). Consider $F:X\rightrightarrows Y$ as a set-valued mapping, and introduce the unconstrained optimization problem where $F$ is the objective

$$(P)\quad \text{minimize } F(x), \text{ subject to } x\in X.$$

The standard Pareto minimality for this problem is stated in the next definition as the efficiency with respect to the partial order $\leq_K$ induced on $Y$ by $K$ on the basis of the equivalence $y_1\leq_K y_2$ iff $y_2-y_1\in K$.

**Definition 1** *A point $(\overline{x},\overline{y})\in\operatorname{Gr}F$ is a Pareto minimum point for $F$, or a Pareto solution for $(P)$, if there exists $\varepsilon\in(0,\infty]$ such that*

$$[F(B(\overline{x},\varepsilon))-\overline{y}]\cap -K=\{0\}. \tag{13}$$

*If $\operatorname{int}K\neq\emptyset$, $(\overline{x},\overline{y})\in\operatorname{Gr}F$ is a weak Pareto minimum point for $F$, or a weak Pareto solution for $(P)$, if there exists $\varepsilon\in(0,\infty]$ such that*

$$[F(B(\overline{x},\varepsilon))-\overline{y}]\cap -\operatorname{int}K=\emptyset.$$

Obviously, in the above definitions, the case $\varepsilon \in (0,\infty)$ corresponds to the local minima, while the case $\varepsilon = +\infty$ describes the global solutions. We mention that the main results of this work apply to both situations.

It is easy to see that $(\overline{x},\overline{y})$ is a minimum for $F$ (in any of the above senses) iff it is a minimum of the same type for the epigraphical set-valued map $\tilde{F} : X \rightrightarrows Y$, $\tilde{F}(x) = F(x) + K$.

Consider now a sequence $(F_n)$ of set-valued mappings acting between $X$ and $Y$. We associate the sequence of optimization problems, with respect to the same order $\leq_K$, as

$$(P_n) \quad \text{minimize } F_n(x), \text{ subject to } x \in X.$$

The main problem we discuss is the following one: having a sequence $(x_n, y_n) \in \operatorname{Gr} F_n$ of Pareto minima for $(P_n)$ (for all $n$) such that $(x_n, y_n) \to (x, y) \in \operatorname{Gr} F$, what can we say about the point $(x, y)$ in relation with problem $(P)$ when $(F_n)$ are, in a sense, approximations of $F$?

A well known fact is that, in general, $(x, y)$ is not a Pareto minimum, even under nice convergence properties of $(F_n)$ towards $F$. Basically, we propose ourselves to describe some general situations when the approximation properties of the sequence $(F_n)$ ensure that $(x, y)$ is a critical point for $(P)$.

We present next the first of the main results, and for this consider the set-valued mappings $(F_n)$, $F$ as the objectives of the problems $(P_n)$ and $(P)$ introduced before.

**Theorem 4** *Suppose that $X, Y$ are Asplund spaces, and take $(x, y) \in \operatorname{Gr} F$. Consider a sequence $(x_n, y_n) \to (x, y)$ such that $(x_n, y_n) \in \operatorname{Gr} F_n$ is a minimum of radius $\varepsilon_n > 0$ for $F_n$ for all $n$. Assume that:*
*(i) $\operatorname{Gr} F$ is locally closed at $(x, y)$;*
*(ii) $K$ is (SNC) at $0$, or $F^{-1}$ is (PSNC) at $(y, x)$;*
*(iii) $\liminf \varepsilon_n > 0$;*
*(iv) there exists a function $\varphi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} \varphi(t) = \varphi(0) = 0$ such that, for all $n$, and for all small $\alpha$*

$$\tilde{F}(B(x,\alpha)) \subset \tilde{F}_n(B(x, \alpha + \varphi(\alpha))). \tag{14}$$

*Then there exists $v^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^*F(x, y)(v^*),$$

*that is, $(x, y)$ is a critical point of $F$.*

The second situation we study uses a new notion of openness, which reads as follows: given a multifunction $F : X \rightrightarrows Y$, a cone $K \subset Y$, a point $(\overline{x},\overline{y}) \in \operatorname{Gr} F$, and two constants $\alpha, \beta > 0$, one says that $F$ is $(\alpha, \beta)-$open with respect to $K$ at $(\overline{x},\overline{y})$ if there exists $\varepsilon > 0$ such that, for any $\rho \in (0, \varepsilon)$,

$$B(\overline{y}, \alpha\rho) \subset F(B(\overline{x}, \rho)) + K \cap B(0, \beta\rho). \tag{15}$$

First, we formulate a result concerning the stability of openness with respect to a cone at Lipschitz perturbations in the global case.

**Theorem 5** *Let $K$ be a closed convex cone, and $F, G : X \rightrightarrows Y$ be two multifunctions such that $\operatorname{Gr} F$ and $\operatorname{Gr} G$ are locally closed. Suppose that $\operatorname{Dom}(F + G)$ is nonempty and let $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ be such that $\alpha > \beta$. If $F$ is $(\alpha, \gamma)-$open with respect to $K$ at every point of its graph, and if $G$ is $\beta-$Aubin at every point of its graph, then $F + G$ is $(\alpha - \beta, \gamma)-$open with respect to $K$ at every point of its graph.*

A similar result, formulated for the local case, holds, which involves the additional property of sum-stability of the pair $(F, G)$ around the reference point.

The main result in this case is given by the next theorem.

**Theorem 6** *Suppose that $X,Y$ are Asplund spaces, and take $(x,y) \in \mathrm{Gr}\, F$. Take $G_n : X \rightrightarrows Y$ and consider a sequence $(x_n, y_n) \to (x,y)$ such that $(x_n, y_n) \in \mathrm{Gr}(F+G_n)$ is a minimum or $F+G_n$ for all $n$. Suppose that:*
*(i) $\mathrm{Gr}\, F$ is locally closed at $(x,y)$ and for all $n$, $\mathrm{Gr}\, G_n$ is locally closed at every point from its graph close to $(x,0)$;*
*(ii) $K$ is (SNC) at $0$ or $F^{-1}$ is (PSNC) at $(y,x)$;*
*(iii) for all $n$, there is $\beta_n > 0$ such that $G_n$ in $\beta_n-$Aubin around every point from its graph close to $(x,0)$, and $\beta_n \to 0$;*
*(iv) for all $n$, the pair $(F, G_n)$ is locally sum-stable around $(x,y,0)$.*
*Then there exists $v^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^*F(x,y)(v^*),$$

*that is, $(x,y)$ is a critical point of $F$.*

**Stage 3.** To study variational problems for optimal control problems.

We consider the following two optimal control problems associated to (1).

**(NO) Norm optimal control problem.** For $T>0$ fixed, minimize the norm of controls $u$ by which the initial point $x$ can be steered to zero in time $T$, i.e., satisfying the equation $V(T)u = S(T)x$.

**(TO) Time optimal control problem.** For $r>0$ fixed, minimize the time needed to drive $x$ to zero in minimum time, by using controls $u \in L^\infty(0,\infty;U)$ satisfying $\|u(s)\| \leq r$ a.e.

First, we considered an abstract problem.

Let $Y$ and $Z$ be two Banach spaces, $G: Y \to Z$ be a linear continuous operator with the adjoint $G^*: Z^* \to Y^*$ and $c \in Y^*$, $c \neq 0$. Suppose that the equation

$$G^* z^* = c \tag{16}$$

has a solution in $Z^*$. This happens if and only if $\gamma(c) > 0$, where

$$\gamma(c) = \inf\{\|Gy\|\, ;\ y \in Y,\ \langle c, y\rangle = 1\}$$

We mention that $\langle y^*, y\rangle$ denotes the image of $y \in Y$ by $y^* \in Y^*$.

Let us define the following minimization/maximization problems:

$$(\mathbf{P_1}):\ \min\left\{\|Gy\|\, ;\ y \in Y,\ \ \langle c, y\rangle = 1\right\},$$

$$(\mathbf{P_2}):\ \max\left\{\langle c, y\rangle\, ;\ y \in Y,\ \ \|Gy\| \leq 1\right\},$$

$$(\mathbf{P_3}):\ \min\left\{\frac{1}{2}\|Gy\|^2 - \langle c, y\rangle\, ;\ y \in Y\right\}$$

and the solvability problem:

$$(\mathbf{P_4}):\ \text{the equation } c \in (G^* \circ J_Z \circ G)(y) \text{ has at least a solution,}$$

where $J_Z : Z \rightrightarrows Z^*$ is the duality map of $Z$.

In the following we shall prove the equivalence between these problems. This result will be used later in the following manner. We shall prove that, in an appropriate setting, **(P3)** has solutions and we will use the fact that **(P4)** has solutions. We point out that the meaning of the fact that **(P3)** has solutions is that there exists $y \in Y$ which minimizes the functional $\frac{1}{2}\|Gy\|^2 - \langle c, y\rangle$.

**Theorem 7** *Assume that the equation (16) has a solution in $Z^*$. Then, the above problems are equivalent, in the sense that if one of them has at least a solution then all of them have at least a solution. Further, $\overline{y}$ is a solution of $(\mathbf{P_3})$ if and only if $\overline{y}$ is a solution of $(\mathbf{P_4})$.*

**Corollary 0.0.2** *Assume that* (**P3**) *has a solution* $y \in Y$*. Then every* $\overline{z}^* \in (J_Z \circ G)(y)$ *with* $G^*\overline{z}^* = c$ *is a minimum norm solution for* (16)*.*

Application to our minimum norm and minimum time control problems

Consider the control system (1). Everywhere in the sequel we assume that the following hypotheses, **(H1)–(H3)**, hold.
**(H1)** $X$ is a reflexive Banach space and $U$ is a Hilbert space.
**(H2)** For every $t > 0$, $V(t) = H(t)^*$ for some $H(t) : X^* \to L^1(0, t; U)$.

**Remark 0.0.3** *Hypothesis* ***(H2)*** *is equivalent to* $V(t)^* x^* \in L^1(0, t; U)$ *for any* $x^* \in X^*$ *and any* $t > 0$*.*

**(H3)** The system (1) is null controllable at any time $t > 0$, i.e.,

$$Range\,(S\,(t)) \subseteq Range\,(V\,(t))\,,\ \forall t > 0. \tag{17}$$

Under **(H3)**, $Range(V(t))$, $t > 0$, is independent of $t$. Denote $R = Range(V(t))$ for some (all) $t > 0$. We introduce on $R$ the following norms:

$$|||r|||_t = \inf\left\{\|u\|\,;\ u \in L^\infty(0, t; U)\,,\ V\,(t)\,u = r\right\},\ t > 0,$$

which define Banach space topologies on $R$. We have that $|||r|||_t \leq |||r|||_s$ for any $t > s > 0$. Therefore, being complete, the norms $|||\cdot|||_t$ , $t > 0$, defined on $R$ are equivalent.

**Remark 0.0.4** *From hypothesis* ***(H3)*** *we get that the control system* (1) *is approximately controllable, i.e.,* $Cl_X(R) = X$*. By* $Cl_X(R)$ *we denoted the closure of* $R$ *with respect to the norm of* $X$*. Indeed, any* $x \in X$ *can be written as* $x = \lim_{\varepsilon\to 0} S(\varepsilon)x$*, hence* $x \in Cl_X(R)$ *since* $S(\varepsilon)x \in R$ *by* ***(H3)****.*

Let

$$\Pi = Cl_R\left(\bigcup_{t>0} Range\,(S\,(t))\right).$$

Here, $Cl_R$ means the closure with respect to the topology of $R$. Clearly, $(\Pi, |||\cdot|||_t)$ is a Banach space.

For any $t > 0$, let $\widetilde{S}\,(t) : X \to \Pi$ be defined by $\widetilde{S}(t)x = S(t)x$ for any $x \in X$. Since the topology on $\Pi$ is stronger than the topology on $X$, by the closed graph theorem, we get that, for any $t > 0$, $\widetilde{S}(t)$ is continuous and we consider his adjoint $\widetilde{S}\,(t)^* : \Pi^* \to X^*$. Here, $\Pi^*$ is the dual space of $\Pi$ with the (equivalent) norms

$$\|p^*\|_t = \sup_{\substack{p\in\Pi \\ |||p|||_t\leq 1}} < p^*, p >,\ t > 0,\ p^* \in \Pi^*.$$

Let us remark that, for any $x^* \in X^*$ and $t > 0$, we have that

$$\|H\,(t)\,x^*\|_{L^1(0,t;U)} = \|x^*\|_{R^*,t}. \tag{18}$$

Here, $R^*$ denotes the dual space of $R$, endowed with the (equivalent) norms

$$\|r^*\|_{R^*,t} = \sup_{\substack{r\in R \\ |||r|||_t\leq 1}} < r^*, r >,\ t > 0,\ r^* \in R^*.$$

Indeed,

$$\begin{aligned} ||H(t)x^*||_{L^1(0,t;U)} &= \sup_{\substack{u\in L^\infty(0,t;U) \\ ||u||\leq 1}} \langle u, H(t)x^*\rangle = \sup_{\substack{u\in L^\infty(0,t;U) \\ ||u||\leq 1}} \langle x^*, V(t)u\rangle \\ &= \sup_{\substack{r\in R \\ |||r|||_t\leq 1}} \langle x^*, r\rangle = \|x^*\|_{R^*,t}. \end{aligned}$$

We will define a new linear bounded operator, denoted $\widetilde{H}\,(T)$, that extends $H(T)$ on $\Pi^*$. To this end we need the following lemma.

**Lemka 0.0.1** *Assume **(H1)–(H3)**. Let $s,t>0$. We have that*

$$(H(s)\widetilde{S}(t)^*p^*)(\tau) = (H(s+\theta)\widetilde{S}(t-\theta)^*p^*)(\tau), \tag{19}$$

*for any $p^* \in \Pi^*$, $\theta \in (0,t)$ and $\tau \in [0,s]$.*

**Remark 0.0.5** *From the above lemma, if we fix $T>0$, for $0<\varepsilon_1<\varepsilon_2$ we have that*

$$(H(T-\varepsilon_1)\widetilde{S}(\varepsilon_1)^*p^*)(\tau) = (H(T-\varepsilon_2)\widetilde{S}(\varepsilon_2)^*p^*)(\tau)$$

*for $\tau \in [0, T-\varepsilon_2]$.*

Fix $T>0$. Now we are able to extend $H(T)$ on $\Pi^*$. We shall denote the extension by $\widetilde{H}\,(T)$. For $p^* \in \Pi^*$ define

$$\left(\widetilde{H}\,(T)\,p^*\right)(t) = \left(H(T-\varepsilon)\widetilde{S}(\varepsilon)^*p^*\right)(t)$$

for $t \in [0, T-\varepsilon]$. By Remark 0.0.5, $\widetilde{H}\,(T)$ is well defined.

**Theorem 8** *Assume **(H1)–(H3)**. For any $p^* \in \Pi^*$ and $T>0$ we have that $\widetilde{H}\,(T)\,p^* \in L^1(0,T;U)$ and $\widetilde{H}\,(T)$ is continuous. Moreover,*

$$\left\|\widetilde{H}\,(T)\,p^*\right\|_{L^1(0,T;U)} \le \|p^*\|_T \tag{20}$$

*for any $p^* \in \Pi^*$.*

We will prove a result (Theorem 9) that will be a key tool in getting on of the main results of this project. To this end, we will need the following two lemmas.

Denote by $Y_T$ the closure in $L^1\,(0,T;U)$ of the functions of type $H\,(T)\,x^*$ with $x^* \in X^*$.

**Lemka 0.0.2** *Assume **(H1)–(H3)**. If $f \in L^1\,(0,T;U)$ and $f|_{[0,\tau]} \in Y_\tau$ for any $\tau < T$, then $f \in Y_T$.*

For any $p^* \in \Pi^*$ define

$$f_{p^*} = \widetilde{H}\,(T)\,p^* \tag{21}$$

which belongs to $L^1\,(0,T;U)$ (see Theorem 8).

**Lemka 0.0.3** *Assume **(H1)–(H3)**. If $V\,(T)\,u = 0$ and $p^* \in \Pi^*$ then*

$$\int_0^T \langle f_{p^*}(s), u(s)\rangle\, ds = 0. \tag{22}$$

The following theorem plays a key role in obtaining the main results of this paper.

**Theorem 9** *Assume **(H1)–(H3)**. For any $p^* \in \Pi^*$ and any $p \in \Pi$ such that $p = V(T)u$ with $u \in L^\infty\,(0,T;U)$, we have that*

$$\langle p^*, p\rangle = \int_0^T \langle f_{p^*}(s), u(s)\rangle\, ds, \tag{23}$$

*where $f_{p^*}$ is given by (21).*

Let $i : \Pi \to \Pi^{**}$ be the canonical injection from $\Pi$ into $\Pi^{**}$, i.e., $\langle i(p), p^*\rangle = \langle p^*, p\rangle$ for any $p \in \Pi$ and any $p^* \in \Pi^*$.

**Corollary 0.0.3** *Assume* ***(H1)–(H3)****. Then*

$$V(T)u = S(T)x$$

*if and only if*

$$\widetilde{H}(T)^{*}u = i(S(T)x).$$

The following result, based on Theorem 9, improves the conclusion of Theorem 8.

**Theorem 10** *Assume* ***(H1)–(H3)****. For any* $p^{*} \in \Pi^{*}$ *we have that*

$$\left\|\widetilde{H}\left(T\right)p^{*}\right\|_{L^{1}(0,T;U)} = ||p^{*}||_{T}. \tag{24}$$

**Remark 0.0.6** *Since* (24) *holds for any* $p^{*} \in \Pi^{*}$*, we get that* $\{\widetilde{H}(T)p^{*};\ p^{*} \in \Pi^{*}\}$ *is a closed subspace of* $L^{1}(0,T;U)$ *(being isometric with* $\Pi^{*}$*). Therefore,* $Y_{T}$ *is a subspace of* $\{\widetilde{H}(T)p^{*};\ p^{*} \in \Pi^{*}\}$*. On another hand, we have proved that* $\widetilde{H}(T)p^{*} \in Y_{T}$ *for any* $p^{*} \in \Pi^{*}$*. Hence,* $Y_{T} = \{\widetilde{H}(T)p^{*};\ p^{*} \in \Pi^{*}\}$*.*

Let $T > 0$ and $x \in X$ be such that $S(T)x \neq 0$. We define the functional

$$\varphi_{x,T}\left(p^{*}\right) = \frac{1}{2}\left\|\widetilde{H}\left(T\right)p^{*}\right\|^{2}_{L^{1}(0,T;U)} - \left\langle p^{*}, S\left(T\right)x\right\rangle,\ p^{*} \in \Pi^{*}.$$

**Theorem 11** *Under* ***(H1)–(H3)****, the minimization problem*

$$\min \varphi_{x,T}\left(p^{*}\right)$$

*has a solution* $\overline{p}^{*}$ *in* $\Pi^{*}$ *and* $\overline{p}^{*} \neq 0$*.*

Characterizations of optimal controls

Now, we are ready to prove the main results of this paper. First, we consider the norm optimal control problem **(NO)**. Let $T > 0$ and $x \in X$ be such that $S\left(T\right)x \neq 0$. By **(H3)**, the equation

$$V\left(T\right)u = S\left(T\right)x \tag{25}$$

has solutions in $L^{\infty}\left(0,T;U\right)$. Our problem is to minimize $||u||$ such that (25) holds.

From Corollary 0.0.3 we have that (25) is equivalent to

$$\widetilde{H}(T)^{*}u = i(S(T)x). \tag{26}$$

Hence, our problem becomes to minimize $||u||$ such that (26) holds.

Denote by *sign* the signum function defined by *sign* $u = u/\left\|u\right\|_{U}$ for $u \in U\backslash\left\{0\right\}$ and *sign* $0 = B_{U}(0,1) := \{u \in U;\ \left\|u\right\|_{U} \leq 1\}$.

**Theorem 12** *Assume* ***(H1)–(H3)****. Let* $\overline{p}^{*} \in \Pi^{*}$ *be a minimizer of* $\varphi_{x,T}$*. Then there exists a minimum norm solution of* (25)*,* $\overline{u} \in L^{\infty}(0,T;U)$*, satisfying*

$$\overline{u}\left(s\right) = \left\|\overline{p}^{*}\right\|_{T}\overline{w}\left(s\right),\ s \in \left[0,T\right], \tag{27}$$

*with*

$$\overline{w}\left(s\right) \in sign\left(\widetilde{H}\left(T\right)\overline{p}^{*}\right)\left(s\right)\ \textit{for a.a.}\ s \in \left[0,T\right]. \tag{28}$$

*Moreover, every solution* $\overline{u}$ *of* (25)*, which satisfies* (27)-(28) *is a minimum norm solution. Consequently,*

$$\mathcal{E}(T,x) = \|\overline{p}^*\|_T \tag{29}$$

*and*

$$\begin{aligned}\left\langle \left(\widetilde{H}\left(T\right)\overline{p}^*\right)(s), \overline{u}(s)\right\rangle &= \sup_{\|v\|_U \leq \mathcal{E}(T,x)} \left\langle \left(\widetilde{H}\left(T\right)\overline{p}^*\right)(s), v\right\rangle \\ &= \mathcal{E}\left(T,x\right)\left\|\left(\widetilde{H}\left(T\right)\overline{p}^*\right)(s)\right\|_U\end{aligned}$$

*for a.a.* $s \in [0,T]$*.*

**Remark 0.0.7** *Expressing the signum function in* (28)*, we obtain that, in the hypotheses of the above theorem, any measurable function* $\overline{u}$ *given by*

$$\overline{u}\left(s\right) \in \begin{cases} \|\overline{p}^*\|_T \dfrac{\left(\widetilde{H}\left(T\right)\overline{p}^*\right)(s)}{\left\|\left(\widetilde{H}\left(T\right)\overline{p}^*\right)(s)\right\|_U}, & \text{if } \left(\widetilde{H}\left(T\right)\overline{p}^*\right)(s) \neq 0 \\ B_U\left(0, \|\overline{p}^*\|_T\right), & \text{if } \left(\widetilde{H}\left(T\right)\overline{p}^*\right)(s) = 0, \end{cases}$$

*where* $\overline{p}^* \in \Pi^*$ *is a minimizer of* $\varphi_{x,T}$*, that satisfies* $V(T)\overline{u} = S(T)x$*, is a minimum norm solution of* (25)*.*

Taking into account the definition of $\widetilde{H}(T)$ we have the following corollary.

**Corollary 0.0.4** *Assume **(H1)–(H3)**. There exists* $\overline{u} \in L^\infty(0,T;U)$ *a minimum norm control for* (25) *given by*

$$\overline{u}(s) \in \mathcal{E}(T,x) sign\ f(s) \text{ a.e. on } [0,T],$$

*for some function* $f \in L^1(0,T;U)$ *such that there exists* $\varepsilon_0 > 0$ *with the property that, for any* $\varepsilon \in (0,\varepsilon_0)$*,*

$$f(s) = (H(T-\varepsilon)x_\varepsilon^*)(s) \text{ a.e. on } [0,T-\varepsilon],$$

*for some* $x_\varepsilon^* \in X^*$*,* $x_\varepsilon^* \neq 0$*.*

**Remark 0.0.8** *Suppose that* $\left(H\left(T-\varepsilon\right)x_\varepsilon^*\right)(t) \neq 0$ *for a.a.* $t \in [0,T-\varepsilon]$*. Then, the minimum norm control is given by*

$$\overline{u}\left(t\right) = \mathcal{E}\left(T,x\right)\frac{\left(H\left(T-\varepsilon\right)x_\varepsilon^*\right)(t)}{\left\|\left(H\left(T-\varepsilon\right)x_\varepsilon^*\right)(t)\right\|_U} \text{ for a.a. } t \in \left[0,T-\varepsilon\right].$$

*Hence,* $\|\overline{u}\left(t\right)\| = \mathcal{E}\left(T,x\right)$ *a.e. on* $[0,T-\varepsilon]$*. Moreover, if* $\left(H\left(T-\varepsilon\right)x_\varepsilon^*\right)(t) \neq 0$ *for a.a.* $t \in \left[0,T-\varepsilon\right]$*, for any* $\varepsilon \in (0,\varepsilon_0)$*, then* $\overline{u}$ *is bang-bang on* $[0,T)$*. This happens, for instance, in case of controls distributed internally, boundary controls.*

**Remark 0.0.9** *Theorem 12 remains valid also in the case when* $U$ *is a reflexive Banach space with the difference that*

$$\overline{w}(s) \in \begin{cases} \dfrac{1}{\|(\widetilde{H}(T)\overline{p}^*)(s)\|_{U^*}} J_U^{-1}((\widetilde{H}(T)\overline{p}^*)(s)), & \text{if } (\widetilde{H}(T)\overline{p}^*)(s) \neq 0 \\ B_U(0,1), & \text{if } (\widetilde{H}(T)\overline{p}^*)(s) = 0. \end{cases}$$

*Here, we denoted by* $J_U$ *the duality map of* $U$*. Since* $U$ *is reflexive,* $J_U^{-1}$ *is the duality map of the dual space* $U^*$*.*

Finally, we consider the time optimal control problem **(TO)**. Using the variational characterization of norm optimal controls (Theorem 12) and the equivalence between the minimum time and minimum norm control problems we get characterizations of the minimum time function and the time optimal controls.

**Theorem 13** *Assume **(H1)**–**(H3)**. Let $r>0$. Then $\mathcal{T}(r,x)$ is characterized by*

$$r=\left\|\widetilde{H}\left(\mathcal{T}(r,x)\right)\overline{p}^{*}\right\|_{L^{1}(0,\mathcal{T}(r,x);U)},$$

*where $\overline{p}^{*}$ is a minimizer of $\varphi_{x,\mathcal{T}(r,x)}$. Furthermore, every*

$$\overline{u}(s)=\left\|\widetilde{H}\left(\mathcal{T}(r,x)\right)\overline{p}^{*}\right\|_{L^{1}(0,\mathcal{T}(r,x);U)}\overline{w}(s), \tag{30}$$

*with*

$$\overline{w}(s)\in sign\left(\widetilde{H}\left(\mathcal{T}(r,x)\right)\overline{p}^{*}\right)(s)\ for\ a.a.\ s\in[0,\mathcal{T}(r,x)], \tag{31}$$

*satisfying $V(T)\overline{u}=S(T)x$, is a time optimal control for $x$. Consequently,*

$$\begin{aligned}\left\langle\overline{u}(s),\left(\widetilde{H}\left(\mathcal{T}(r,x)\right)\overline{p}^{*}\right)(s)\right\rangle&=\sup_{\|v\|\le r}\left\langle v,\left(\widetilde{H}\left(\mathcal{T}(r,x)\right)\overline{p}^{*}\right)(s)\right\rangle\\&=r\left\|\left(\widetilde{H}\left(\mathcal{T}(r,x)\right)\overline{p}^{*}\right)(s)\right\|_{U}\end{aligned}$$

*for a.a. $s\in[0,\mathcal{T}(r,x)]$.*

In the frame of this research project we also studied the implications of a well-known metric inequality condition on sets to the applicability of standard necessary optimality conditions for constrained optimization problems when a new constraint is added. We compared this condition with several other constraint qualification conditions in literature and, due to its metric nature, we applied it to nonsmooth optimization problems in order to perform first a penalization and then to give optimality conditions in terms of generalized differentiability.

Consider the basic optimization problem

$$\min f(x),\ \text{subject to}\ g(x)\le 0,$$

and let $\overline{x}\in X$ be an optimal solution of this problem. The first-order necessary optimality condition is

$$\nabla f(\overline{x})(u)\ge 0,\ \forall u\in T_B(M_g,\overline{x}), \tag{32}$$

where

$$M_g:=\{x\in X\mid g(x)\le 0\}$$

is the set of feasible points.

If the constraint is active at $\overline{x}$, i.e., $g(\overline{x})=0$, we have to suppose that $\nabla g(\overline{x})\neq 0$ in order to obtain that

$$T_B(M_g,\overline{x})=T_U(M_g,\overline{x})=\operatorname{cl}T_{DM}(M_g,\overline{x})=\{u\in X\mid\nabla g(\overline{x})(u)\le 0\}. \tag{33}$$

Now condition (32) becomes

$$\nabla f(\overline{x})(u)\ge 0,\ \text{subject to}\ \nabla g(\overline{x})(u)\le 0,$$

which can be seen as the fact that $0\in X$ is an optimal solution to the linear problem

$$\min\nabla f(\overline{x})(u),\ \text{subject to}\ \nabla g(\overline{x})(u)\le 0.$$

Then, since for linear problems there is no need of supplementary qualification conditions for applying Karush-Kuhn-Tucker Theorem, we get $\lambda \geq 0$ such that

$$\nabla f(\overline{x}) + \lambda \nabla g(\overline{x}) = 0.$$

For the scalar function $g$, the condition $\nabla g(\overline{x}) \neq 0$ is equivalent to the well known Mangasarian-Fromowitz constraint qualification condition.

We continue by adding a new scalar inequality constraint. Therefore, suppose that the constraint is expressed in the same way, but with a function $g = (g_1, g_2) : X \to \mathbb{R}^2$. Let $\overline{x}$ be a feasible point. In the case of active constraints, that is, $g(\overline{x}) = 0 \in \mathbb{R}^2$, the Mangasarian-Fromowitz condition is: there exists $u \in X$ such that $\nabla g_1(\overline{x})(u) < 0$ and $\nabla g_2(\overline{x})(u) < 0$. On the same lines as before, this condition ensures that

$$T_B(M_g, \overline{x}) = T_U(M_g, \overline{x}) = \operatorname{cl} T_{DM}(M_g, \overline{x}) = \{u \in X \mid \nabla g_1(\overline{x})(u) \leq 0, \nabla g_2(\overline{x})(u) \leq 0\}.$$

In particular, this means that

$$T_B(M_g, \overline{x}) = T_B(M_{g_1}, \overline{x}) \cap T_B(M_{g_2}, \overline{x})$$

and, in fact, this is the essential relationship to get, on the same argument as in the case of a single scalar-valued constraint, that there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$\nabla f(\overline{x}) + \lambda_1 \nabla g_1(\overline{x}) + \lambda_2 \nabla g_2(\overline{x}) = 0.$$

Therefore, every time we consider another scalar-valued constraint which is active at the underlying point $\overline{x}$, one has to impose the Mangasarian-Fromowitz condition, and this condition is stronger than the Mangasarian-Fromowitz conditions for every of the components of $g$.

Our aim is to present a situation when one can replace the general Mangasarian-Fromowitz condition with Mangasarian-Fromowitz conditions for each coordinate function, under a supplementary hypothesis.

**Theorem 14** *Let $X$ be a normed vector space and $M_1, M_2 \subset X$ be closed sets. Take $\overline{x} \in M_1 \cap M_2$. Suppose that the following regularity assumption holds: there exist $s > 0, \mu > 0$ such that for all $x \in B(\overline{x}, s) \cap M_1$,*

$$d(x, M_1 \cap M_2) \leq \mu d(x, M_2). \tag{MC}$$

*Then*

$$\begin{array}{l} T_B(M_1, \overline{x}) \cap T_U(M_2, \overline{x}) \subset T_B(M_1 \cap M_2, \overline{x}) \\ T_U(M_1, \overline{x}) \cap T_B(M_2, \overline{x}) \subset T_B(M_1 \cap M_2, \overline{x}) \\ T_U(M_1, \overline{x}) \cap T_U(M_2, \overline{x}) = T_U(M_1 \cap M_2, \overline{x}). \end{array} \tag{CHIP}$$

Recall that a function $f : X \to Y$ is metrically subregular at $(\overline{x}, f(\overline{x}))$ with respect to $M \subset X$ when $\overline{x} \in M$ and there exist $s > 0$, $\mu > 0$ such that for every $u \in B(\overline{x}, s) \cap M$

$$d(u, f^{-1}(f(\overline{x})) \cap M) \leq \mu \left\| f(\overline{x}) - f(u) \right\|.$$

Next we present the equivalence (up to a change of the involved constants) of several metric conditions.

**Proposition 0.0.3** *Take $\overline{x} \in M_1 \cap M_2$. The next assertions are equivalent:*

**Proposition 0.0.4** *(i) there exist $s, \mu > 0$ such that for all $x \in B(\overline{x}, s) \cap M_1$,*

$$d(x, M_1 \cap M_2) \leq \mu d(x, M_2).$$

*(ii) there exist $r, t, \mu > 0$ such that for all $x \in B(\overline{x}, r) \cap M_1$,*

$$d(x, M_1 \cap M_2) \leq \mu d(x, B(\overline{x}, t) \cap M_2).$$

*(iii) there exist $r, \nu > 0$ such that for all $x \in B(\overline{x}, r) \cap M_2$,*

$$d(x, M_1 \cap M_2) \leq \nu d(x, M_1).$$

*(iv) there exist $r, \nu > 0$ such that for all $x \in B(\overline{x}, r)$,*

$$d(x, M_1 \cap M_2) \leq \nu \left( d(x, M_1) + d(x, M_2) \right). \tag{MI}$$

*(v) the function $g : X \times X \to X$ given by $g(x, y) := x - y$ is metrically subregular at $(\overline{x}, \overline{x}, 0)$ with respect to $M_1 \times M_2$.*

*(vi) the function $h : X \times X \to \mathbb{R}$ given by $h(x, y) := d(x, y)$ is metrically subregular at $(\overline{x}, \overline{x}, 0)$ with respect to $M_1 \times M_2$.*

Combining the previous results, we present a consequence for the systems with multiple constraints.

**Corollary 0.0.5** *Let $g = (g_1, g_2) : X \to \mathbb{R}^2$ be differentiable, consider $\overline{x} \in X$ such that $g_1(\overline{x}) = g_2(\overline{x}) = 0$ and $\nabla g_1(\overline{x}) \neq 0, \nabla g_2(\overline{x}) \neq 0$. If there exist $s > 0, \mu > 0$ such that for all $x \in B(\overline{x}, s) \cap M_{g_1}$,*

$$d(x, M_{g_1} \cap M_{g_2}) \leq \mu d(x, M_{g_2}),$$

*then*

$$T_B(M_g, \overline{x}) = \left\{ u \in X \mid \nabla g(\overline{x})(u) \leq 0 \right\}.$$

Let us consider a general constraint system $h(x) \leq 0$, where $h : X \to \mathbb{R}^n$ is a $C^1$ function and $h(x) \leq 0$ means that $h_i(x) \leq 0$ for all $i \in \overline{1, n}$. The Mangasarian-Fromowitz constraint qualification condition at $\overline{x}$ when $h$ is active is

$$\exists u \in X : \nabla h(\overline{x})(u) < 0. \tag{MFCQ}$$

It is also well known that (MFCQ) is equivalent to the metric regularity around $(\overline{x}, 0)$ of the set-valued map $\tilde{h} : X \rightrightarrows \mathbb{R}^n$ given by

$$\tilde{h}\,(x) := h\,(x) + \mathbb{R}^n_+.$$

We denote this metric regularity condition by (MRCQ).

There is an important amount of literature that underlines the idea that metric subregularity still ensures the validity of many facts in optimization and, in particular, can replace metric regularity in qualification conditions. For instance, in the context we discuss here, the metric subregularity qualification condition (MSCQ) amounts to say that the above set-valued map is metrically subregular at $(\overline{x}, 0)$.

Consider next a $C^1$ function $g = (g_1, g_2) : X \to \mathbb{R}^2$, the associated constraint system of inequalities $g(x) \leq 0$, and let $\overline{x}$ be a feasible point where both scalar constraints are active. Let us discuss the relationship between (MFCQ), (MI) and (MSCQ).

**Proposition 0.0.5** *In the above notation we have the following implications:*

**Proposition 0.0.6** *(i)* (MFCQ) $\Leftrightarrow$ (MRCQ) $\Rightarrow$ (MSCQ);
*(ii)* (MSCQ) $\Rightarrow$ (MI);
*(iii)* [(MSCQ) *for* $g_1$ *and* $g_2$] + (MI) $\Rightarrow$ (MSCQ) *for* $(g_1, g_2)$.

Next, we present some consequences of the metric condition discussed above and we give as well some other similar conditions that can be used in various contexts.

Consider a scalar function $f : X \to \mathbb{R}$ and recall that the Hadamard upper directional derivative of $f$ at $\overline{x} \in X$ in direction $u \in X$ is

$$d_+ f(\overline{x}, u) = \limsup_{u' \to u, t \downarrow 0} t^{-1}(f(\overline{x} + tu') - f(\overline{x})),$$

while the Hadamard lower directional derivative of $f$ at $\overline{x}$ in direction $u$ is

$$d_- f(\overline{x}, u) = \liminf_{u' \to u, t \downarrow 0} t^{-1}(f(\overline{x} + tu') - f(\overline{x})).$$

**Definition 2** *Let $f : X \to \mathbb{R}$ be a function and $A \subset X$, $L \subset S(0,1)$ be nonempty closed sets. One says that $\overline{x} \in A$ is a local directional minimum point for $f$ on $A$ with respect to (the set of directions) $L$ if there exists a neighborhood $U$ of $\overline{x}$ such that for every $x \in U \cap A \cap (\overline{x} + \operatorname{cone} L)$, $f(\overline{x}) \leq f(x)$.*

**Proposition 0.0.7** *Let $f : X \to \mathbb{R}$ be a function, $A \subset X$, $L \subset S(0,1)$ be nonempty closed sets and $\overline{x} \in A$. Suppose that there exist $s > 0, \mu > 0$ such that*

$$\forall x \in B(\overline{x}, s) \cap A : d(x, A \cap (\overline{x} + \operatorname{cone} L)) \leq \mu d(x, \overline{x} + \operatorname{cone} L). \tag{34}$$

*(i) If $\overline{x}$ is a local directional minimum point for $f$ on $A$ with respect to $L$, then $d_+ f(\overline{x}, u) \geq 0$ for all $u \in T_B(A, \overline{x}) \cap L$.*

*(ii) Moreover, if $X$ is finite dimensional and $d_- f(\overline{x}, u) > 0$ for all $u \in T_B(A, \overline{x}) \cap L$, then there exists $\alpha > 0$ such that $\overline{x}$ is a local directional minimum point for $f(\cdot) - \alpha \|\cdot - \overline{x}\|$ on $A$ with respect to $L$.*

An interesting fact is that metric conditions of the type (MC) come into play as weak assumptions to ensure the validity of some penalization principles. Even if all the conditions in Proposition 0.0.3 are equivalent, the change of constants is important in the construction of the penalty function. More details will be given after the next results.

**Theorem 15** *Let $f : X \to \mathbb{R}$ be a function and $A, B \subset X$ be nonempty, closed sets. Let $\overline{x} \in A \cap B$ be a local minimum point for $f$ on $A \cap B$. Suppose that*

**Theorem 16** *(i) there exist $\varepsilon > 0$ and $\ell > 0$ such that $f$ is $\ell-$Lipschitz on $B(\overline{x}, \varepsilon)$;*

*(ii) there exist $s > 0, \mu > 0$ such that for all $x \in B(\overline{x}, s) \cap A$,*

$$d(x, A \cap B) \leq \mu d(x, B).$$

*Then $\overline{x}$ is a local minimum point for $f + \ell \mu d(\cdot, B)$ on $A$ and a local minimum point (without constraints) for*

$$f + \ell (1 + \mu) d(\cdot, A) + \ell \mu d(\cdot, B). \tag{35}$$

We apply this generalized penalty result for getting necessary optimality conditions in the dual space for directional minima.

**Corollary 0.0.6** *Let $X$ be an Asplund space, $f : X \to \mathbb{R}$ be a function, $A \subset X$, $L \subset S(0,1)$ be nonempty closed sets and $\overline{x} \in A$. Suppose that:*

**Corollary 0.0.7** *(i) $\overline{x}$ is a local directional minimum point for $f$ on $A$ with respect to $L$;*

*(ii) there exist $\varepsilon > 0$ and $\ell > 0$ such that $f$ is $\ell-$Lipschitz on $B(\overline{x}, \varepsilon)$;*

*(iii) there exist $s > 0, \mu > 0$ such that for all $x \in B(\overline{x}, s) \cap A$,*

$$d(x, A \cap (\overline{x} + \operatorname{cone} L)) \leq \mu d(x, \overline{x} + \operatorname{cone} L).$$

*Then there are $u^* \in N(A, \overline{x}), v^* \in N(\operatorname{cone} L, 0)$ with $\|u^*\| \leq \ell(1+\mu)$ and $\|v^*\| \leq \ell\mu$, such that*

$$-u^* - v^* \in \partial f(\overline{x}).$$

In the next result we point out that a metric condition can be imposed as well for a functional constraint in order to get a penalty result. Let $Z$ be a normed vector space, $g : X \to Z$ be a function and $Q \subset Z$ be a pointed closed convex cone. As above, one defines the set-valued map $\tilde{g} : X \rightrightarrows Z$ given by $\tilde{g}(x) := g(x) + Q$ and one considers $g^{-1}(-Q) = \tilde{g}^{-1}(0)$ as the feasible set.

**Theorem 17** *Let $\overline{x} \in g^{-1}(-Q)$ be a local minimum of $f$ on $g^{-1}(-Q)$. Suppose that*

**Theorem 18** *(i) there exist $\varepsilon > 0$ and $\ell > 0$ such that $f$ is $\ell-$Lipschitz on $B(\overline{x}, \varepsilon)$;*

*(ii) there exist $s, \mu > 0$ such that for all $x \in g^{-1}(-Q + B(0, s)) \cap B(\overline{x}, s)$*

$$d(x, g^{-1}(-Q)) \leq \mu d(0, \tilde{g}(x) \cap B(0, s)).$$

*Then $(\overline{x}, 0)$ is a local minimum for the function $(x, z) \mapsto f(x) + \ell\mu \|z\| + \ell(1+\mu) d((x, z), \operatorname{Gr} \tilde{g})$.*

*If, moreover, $X, Z$ are Asplund spaces, then*

$$-\partial f(\overline{x}) \cap D(0, \ell(1+\mu)) \cap D^*\tilde{g}(\overline{x}, 0)(Q^+ \cap D(0, \ell\mu)) \neq \emptyset.$$

Finally, we give some optimality conditions for a concept of directional minimum with respect to two sets of directions for vectorial functions.

**Lemka 0.0.4** *Let $X, Y$ be Banach spaces, $f : X \to Y$ a continuously differentiable function, $M \subset S_Y$ a closed and nonempty set, and $\overline{x} \in X$. Suppose that one of the following sets of conditions holds:*

**Lemka 0.0.5** *(i)* $\operatorname{cone} M$ *is convex and there exists $u_0 \in X$ such that $\nabla f(\overline{x})(u_0) \in -\operatorname{int}(\operatorname{cone} M)$;*

*(ii) the map $g : X \times Y \to Y$ given by $g(x, y) = f(x) - y$ is metrically subregular at $(\overline{x}, f(\overline{x}), 0)$ with respect to $X \times f^{-1}(f(\overline{x}) - \operatorname{cone} M)$.*

*Then*

$$T_B(f^{-1}(f(\overline{x}) - \operatorname{cone} M), \overline{x}) = \nabla f(\overline{x})^{-1}(-\operatorname{cone} M). \tag{36}$$

**Definition 3** *Let $X$ and $Y$ be normed vector spaces, $K \subset Y$ a closed ordering cone with nonempty interior, $f : X \to Y$, and $A \subset X$, $L \subset S_X$, $M \subset S_Y$ closed sets. One says that $\overline{x} \in A$ is a weak local directional Pareto minimum point for $f$ with respect to the sets of directions $L$ and $M$ on $A$ if there exists a neighborhood $U$ for $\overline{x}$ such that*

$$[[f(A \cap U \cap (\overline{x} + \operatorname{cone} L)) \cap (f(\overline{x}) - \operatorname{cone} M)] - f(\overline{x})] \cap (-\operatorname{int} K) = \emptyset.$$

We present now the necessary optimality conditions.

**Theorem 19** *Let $X$, $Y$ be Banach spaces, $f : X \to Y$ a continuously differentiable function, $K \subset Y$ a closed convex ordering cone with nonempty interior, $A \subset X$, $L \subset S_X$ and $M \subset S_Y$ closed and nonempty sets, and $\overline{x} \in A$. Assume there exist $s, \mu, t, \gamma > 0$ such that*

$$d(x, A \cap (\overline{x} + \operatorname{cone} L)) \leq \mu d(x, \overline{x} + \operatorname{cone} L), \ \forall x \in B(\overline{x}, s) \cap A,$$

*and*

$$d(x, A \cap (\overline{x} + \text{cone}L) \cap f^{-1}(f(\overline{x}) - \text{cone}M)) \leq \gamma d(x, f^{-1}(f(\overline{x}) - \text{cone}M)),$$
$$\forall x \in B(\overline{x}, t) \cap A \cap (\overline{x} + \text{cone}\,L).$$

*Suppose also that either (i) or (ii) from Lemma 0.0.4 hold, and $\overline{x}$ is a weak local directional Pareto minimum for $f$ on $A$ with respect to $L$ and $M$.*

*Then*

$$\nabla f(\overline{x})(u) \notin -\operatorname{int} K,\ \forall u \in T_B(A, \overline{x}) \cap \text{cone}L \cap \nabla f(\overline{x})^{-1}(-\text{cone}M).$$

In the frame of this research project we have **published 5 articles (in ISI journal)**, financially supported by the project, confirmed by the corresponding text from the *Acknowledgement* section:

1. S. Bilal, O. Carja, T. Donchev, A. I. Lazu, Nonlocal problem for evolution inclusions with one-sided Perron nonlinearities, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 113 (3), 1917-1933, 2019. DOI: 10.1007/s13398-018-0589-6.

2. O. Cârjă, A. I. Lazu, Minimum time and minimum energy for linear control systems, Systems and Control Letters 137, 2020. https://doi.org/10.1016/j.sysconle.2020.104629

3. O. Cârjă, A. I. Lazu, Minimum time and minimum energy for linear systems; a variational approach, Appl Math Optim (2020). https://doi.org/10.1007/s00245-020-09715-x

4. T. Donchev, S. Bilal, O. Cârjă, N. Javaid, A. I. Lazu, Evolution inclusions in Banach spaces under dissipative conditions, Mathematics 2020, 8(5), 750. https://doi.org/10.3390/math8050750

5. Durea, M., Strugariu, R. On the sensitivity of Pareto efficiency in set-valued optimization problems. J Glob Optim 78, 581-596 (2020). https://doi.org/10.1007/s10898-020-00925-9

Also, we have **submitted for publication one paper in an ISI journal** and one paper is in preparation (A. I. Lazu, V. Postolache).

6. M. Durea, D. Maxim, R. Strugariu, Metric inequality conditions on sets and consequences in optimization, submitted.

The research results obtained in the frame of this project were disseminated at the following **international conferences**: ROMFIN&FSDONA, June 10-15, 2019, Turku, Finland (A. I. Lazu, "*Nonlocal m-dissipative evolution inclusions in general Banach spaces* "), EQUADIFF Conference, July 8-12, 2019, Leiden, Netherlands (A. I. Lazu, "*On the equivalence of minimum time and minimum norm control*"), International conference on Continuous Optimization (ICCOPT 2019), August 3-8, 2019, Berlin, Germany (R. Strugariu, "*Stability of the directional regularity*"), The Fifth Conference of the Mathematical Society of the Republic of Moldova dedicated to the 55th anniversary of the foundation of the Vladimir Andrunachievici Institute of Mathematics and Computer Science - IMCS55, September 28 - October 1, 2019, Chisinau, Republic of Moldova (R. Strugariu, "*On directional stability of mappings*"), International Conference on Applied and Pure Mathematics, Octorber 31 – November 3, 2019, Iaşi (A. I. Lazu, "*Minimum time and minimum energy for linear systems*") and at "Al. I. Cuza" University Conference "Dies Academici" 30 octombrie 2020 (A. I. Lazu, "*On the minimum norm control for linear systems*").

Project leader,
dr. Alina Ilinca LAZU